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A TWO-PERIOD REPAIR PARTS INVENTORY MODEL
FOR A NAVAL AIR REWORK FACILITY

by

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projected production information in calculating RSS inventory levels from one to two periods, and compares the expected total costs from both systems under the assumption of a binomial demand distribution which is appropriate to a NARF. As a result of this comparison, the conclusion is made that the two-period model offers only very minor expected cost advantages over a single-period formulation, while also being much more difficult to utilize due to the complex calculations involved in the computation process.

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A Two-Period Repair Parts Inventory Model for a Naval Air Rework Facility

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TABLE OF SYMBOLS

k	optimal starting inventory quantity pertinent to the second quarter of a two-period model.
m,n	production schedules for particular periods.
p	historical probability that a given repair part will be replaced during overhaul of its parent equipment; associated with the binomial distribution.
s	possible demand value during the second quarter of a dual-period model.
u	possible demand value during the first period of either a single or multi-period model.
x	inventory balance brought forward from a previous period.
y	starting inventory quantity for the initial production period.
y*	optimal opening inventory balance for the first quarter of a single or multi-period model.
C	unit price for a specified repair part.
H	surplus cost rate for a specified repair part.
P	shortage cost rate for a specified repair part.
f(x;g)	optimal expected total variable costs for a period; associated with a previous period closing balance of x and a demand distribution of g.
g(u)	demand distribution values during a specified period.
L(y;g)	expected variable costs of placing a quantity, y, in inventory at the start of one period, given a demand probability distribution of g.

I. INTRODUCTION

A. BACKGROUND

Naval Air Rework Facilities (NARFs) perform overhaul and maintenance actions on various components utilized by fleet aviation units. These rework activities are accomplished in accordance with a quarterly production schedule which is established by a joint workload conference between the NARF and the Naval Air Logistics Command. Information on anticipated NARF workload requirements is also available for several additional future periods. Accordingly, the NARFs are in the process of implementing a Material Requirements Planning (MRP) system which will utilize the available forecast to project requirements for individual spare parts used in the rework process. By establishing such a system, the NARF will be able to accomplish its assigned mission more efficiently by reducing tne number of work stoppages caused by stockouts [Ref. 1: 10].

MRP systems are intended to reduce or even eliminate repair part inventory requirements through the technique of phasing item arrivals to coincide with their need within the production process. However, variations in procurement lead-time and in the actual production schedules necessitate the establishment of some form of backup inventory support. For this purpose, the Naval Supply Center (NSC) supporting the

NARF has established a Ready Supply Store (RSS) of parts which are anticipated to be used during the production process. Once located in the RSS, these parts are protected from issue to other NSC customers. Consequently, factors such as the need to maintain maximum visibility of critical aviation parts within the entire Navy Supply System, the limitations on funding available to the RSS, and the requirement to support customers other than the NARF mandate that the range and depth of items within the RSS inventory be carefully monitored.

Prior to development of the MRP system, RSS stock quantities were established by the traditional method of using historical demand data to compute appropriate levels for individual items. This technique is effective in situations where the workload is reasonably constant from one period to the next, but can result in large surplus or shortage quantities when the production schedule changes to any great extent. Since component overhaul activity at the NARF is susceptible to quarterly fluctuations, a system which determines phased repair part requirements based on anticipated production schedules offers significant advantages over one founded solely on demand history. Accordingly, an appropriate single-period inventory model utilizing the known production information for the next quarter was proposed by McMasters [Ref. 2: 4-15]. This model distinguishes between items which are replaced 100% of the time during overhaul and

those which are not, focusing on the latter category since the former should necessarily be stocked in quantities allowing for total replacement. The ensuing inventory system is then based on obtaining a balance between expected shortage and surplus costs in determining optimal stock quantities.

B. PURPOSE

While the McMasters single-period model represents a significant step forward by utilizing projected future requirements in place of historical demand, it does not make maximum use of available production forecasts since only the workload for the next quarter is considered. In particular, surplus costs projected at the end of a given period on a specific item may be greatly reduced by considering the anticipated demand for that item in future periods. Alternately, a significant demand forecast in ensuing periods will probably result in a decreased tendency on the part of the system to accept a potential shortage of that item in the initial period. A model which incorporates anticipated rework activity beyond the upcoming production period should provide a more accurate reflection of total expected costs over a particular length of time and, thus, offers the potential of creating a more cost effective inventory mix.

Accordingly, the objectives of this thesis are to:

1. Develop a model which utilizes available production schedules for the next two quarters in determining appropriate inventory quantities for the supporting RSS.

2. Conduct a cost comparison between the optimal results obtained from single and dual-period models.

Completion of this two-period inventory system will represent the first step toward development of a multi-period model which considers all available production information and covers a full procurement leadtime for each RSS stock item.

C. PREVIEW

Chapter II describes the general model utilized for the duration of this paper and its behavior when used in a single-period analysis. In Chapter III, this basic model is used as a two-period inventory system whose performance is examined under the assumption of a uniform demand distribution. Chapter IV describes the two-period model under the assumption of a binomial demand distribution. This distribution provides the best representation of demand appropriate to a NARF [Ref. 2: 4]. A sensitivity analysis on the results obtained from using the two-period binomial model is conducted in Chapter V. Finally, Chapter VI provides a summary of results as well as recommendations for future actions.

II. THE SINGLE-PERIOD MODEL

A. PRELIMINARY INFORMATION

An inventory model compatible with the general ideas discussed in the first chapter must possess several essential features. These are:

1. The model should select a stock quantity for each specific item which minimizes inventory-related costs.

2. As with the McMasters single-period system, the model should evaluate the trade-offs between expected surplus and shortage costs in computing appropriate inventory levels.

3. The model should be able to incorporate two or more production periods in the planning horizon.

4. The model should possess the capability of dealing with production schedules which vary from one period to the next. In other words, it must be able to treat demand probability distributions which have different parameter values in successive periods.

The general system described in the remainder of this thesis possesses the essential features detailed above and is an adaptation of a model developed by Karlin [Ref. 3: 231-243].

B. GENERAL ASSUMPTIONS

Before proceeding to examine the single-period model, it is necessary to specify the general assumptions which have been made in its formulation. These assumptions are listed as follows:

1. No backorders are permitted. If the required spare is not available from RSS stock when the demand for it occurs, then that demand is assumed to be filled from

outside the RSS and the shortage penalty cost reflects the time required to obtain a unit from the external supply system.

2. The procurement leadtime for orders of replenishment stock is assumed to be zero. This permits the optimal stock quantity computed during the review cycle at the end of one period to be immediately on hand at the beginning of the next period. This assumption is fairly realistic for items carried at the NARF which are also maintained in stock at the supporting NSC.

3. The unit rates for ordering, surplus and shortage costs are treated as linear. Thus, for example, a cumulative excess of supply over demand of 3 units at the end of a particular period will result in a surplus penalty which is three times higher than if the total excess quantity was only 1.

C. COST ELEMENTS

There are many types of costs associated with maintaining inventory systems, ranging from obvious factors such as procurement and storage costs to less evident ones like the cost of operating data gathering procedures for the system [Ref. 4: 10]. However, the only costs which need to be considered in calculating the optimal inventory quantity for a particular item are those which vary with the quantity ordered. The components of the total variable cost (TVC) associated with the model developed in this thesis are discussed in the following paragraphs.

1. Ordering Costs

A basic assumption behind the MRP concept is that an order review will be conducted once per production period to determine if a stock replenishment action for each individual line item should be accomplished. Since this review will be

conducted at the close of each period for all items and since subsequent requisitions will be created as part of the review process, the cost of accomplishing the review and submitting the requisitions will be considered constant from period to period. Thus, the only variable cost associated with the ordering process will be the cost of the quantity procured. This is the product of the unit price for a particular item, C, and the quantity of that item which is ordered.

2. Surplus Costs

Surplus costs are those associated with having a quantity of an item on hand in inventory beyond the period in question. They can thus be expressed as the cumulative excess of supply over demand at the conclusion of a given period times a constant surplus cost rate per item, H. Since the costs of storing RSS stock remain relatively unchanged from period to period, the surplus cost rate can be treated as a penalty paid whenever the supply system is unwilling to take back NARF excess stock on a full credit basis. This penalty can be quantified by combining the unit price of the item in question with the historical percentage of excess NARF material accepted by the supply system for credit. Accordingly, surplus cost rates which are less than a particular item's unit price will be utilized in evaluating specific model results.

3. Shortage Costs

Whenever a rework requirement for a spare part cannot be filled from the RSS inventory, a shortage cost representing the penalty associated with a work stoppage is incurred. This shortage cost rate per item, P, is assessed at the conclusion of a period on the cumulative excess of demand over supply. The shortage cost is typically the most difficult parameter in an inventory model to quantify since a reasonable estimate must include an accumulation of all production-related delay costs such as the labor charge to backrob or place the end item in storage until the required part arrives [Ref. 2: 32]. As a result, particular emphasis will be placed on examining the sensitivity of results obtained from specific models to changes in the shortage cost rate.

D. SINGLE-PERIOD OBJECTIVE FUNCTION

By utilizing the Karlin inventory model as adapted to include particular NARF costs, an objective function expressing TVC for a single-period model can be formulated. This function can then be used to identify that inventory order quantity for a particular item which results in the lowest TVC for a given demand probability distribution. As previously noted, only those items which are replaced less than 100% of the time during overhaul need to be examined in the model.

Using the surplus and shortage cost parameters as previously discussed, the expected costs at the end of one

period given an initial quantity of stock, y, and a continuous demand probability distribution, g(u), may be expressed as L(y;g), where:

$$L(y;g) = H\int_{0}^{y}(y-u)g(u)du + P\int_{y}^{\infty}(u-y)g(u)du. \qquad (1)$$

On the interval where demand, u, takes on values from 0 to y, the amount (y-u) represents a surplus quantity. Similarly, the amount (u-y) is a shortage quantity whenever u ranges from y to infinity. Therefore, the first term of L(y;g) is the product of the surplus cost rate per unit and the expected number of surplus units at the end of the period, while the second term consists of the product of the shortage cost rate and the expected number of units short at the end of the period.

The TVC incurred when x is the amount of an item on hand at the conclusion of the previous period and y is the quantity in stock at the beginning of the present period can be calculated as:

$$TVC_y = C[y-x] + L(y;g)$$

where C[y-x] represents the total procurement costs and L(y;g) was given by equation (1). The minimum TVC for a one-period model will be denoted f(x;g), where:

$$f(x;g) = \min_{y\geq x}\{C[y-x] + L(y;g)\}. \qquad (2)$$

The optimal y quantity used in identifying f(x;g) will be abbreviated as y*.

E. DEMAND PROBABILITY DISTRIBUTIONS

As discussed in the previous section, the value of y* obtained from a specific model is heavily dependent upon the applicable demand probability distribution. A realistic expression of actual demand experienced at a NARF may be derived by using the binomial distribution. Under this approach, a particular repair part contained in a given component will either be required or not required in any rework action, and the probability, p, of the part being required can be estimated using historical data available at the NARF. Given that the production schedule for a particular component is n during the period in question, and that one unit of a certain repair part is contained in that component, the probability, g(u), that total demand for that repair part during this period will be u units may be expressed as [Ref. 5: 84],

$$g(u) = \frac{n!}{u!(n-u)!} p^{u}(1-p)^{n-u}.$$

Recursion equations may be utilized to simplify the process of computing the entire range of demand for this discrete function. Thus, it can be easily shown that:

$$g(u) = \begin{cases} (1-p)^n \text{ for } u = 0 \\ \\ \frac{[n-(u-1)]}{u(1-p)} \, pg(u-1) \text{ for } u > 0 \end{cases} \qquad (3)$$

The binomial demand distribution as used above is appropriate only for the situation in which a quantity of one of the repair part in question is present in the parent equipment being overhauled. The case involving more than one of a specific part in each of identical end items can also be shown to be binomial with an additional parameter to describe the multiplicity of the part [Ref. 2: 4]. However, when a common item is present in several different equipments, its demand distribution is not binomial and will not be considered further in this thesis.

The binomial distribution is cumbersome to use in identifying optimal inventory quantities through the Karlin model. Thus, in order to illustrate the optimization process, the initial problem formulation will utilize the uniform distribution to represent predicted demand. Both continuous and discrete uniform distribution functions will be considered.

Assume that the production schedule for a particular component during the period in question is n, and that one unit of a specified repair part is present in that component, but that not all of it needs to be replaced. An example of this would be adding additional oil to a partially full hydraulic reservoir. Then, all values of demand, u, between 0 and n

are equally likely under the continuous uniform distribution [Ref. 6: 113-114]. By restricting the range of possible demand values to integer quantities, the discrete uniform distribution may be applied. In this situation, the probability, g(u), that total demand for the repair part in question will be u units, where u = 0,1,2,...,n, is given in general by:

$$g(u) = \begin{cases} \frac{1}{n} \text{ for } 0 \leq u \leq n \\ \\ 0 \text{ otherwise.} \end{cases} \qquad (4)$$

F. SINGLE-PERIOD DERIVATIONS

In this section, the specific single-period models associated with the continuous and discrete uniform demand distributions as well as the binomial density function will be derived. For all models, the optimal stock quantity, y*, at the beginning of the period, given a production schedule of n units and a previous period ending balance of x, will be that value of y which minimizes the expected TVC. The model utilizing a continuous uniform demand distribution function will allow the y* value to be calculated explicitly using the calculus. However, the models based on discrete distributions require the approach of finite differences which involves comparison of TVC values associated with integer inventory quantities.

The optimal inventory value computed by applying each model should be viewed both as a critical number which acts

as a cutoff point for determining whether an order should be placed as well as a high limit defining what the order quantity should be. Specifically, if y* is less than or equal to the quantity, x, on hand at the close of a given period when the level review is accomplished, then the optimal policy is not to order additional stock. Conversely, a y* value greater than the closing inventory balance implies that a requisition for (y*-x) is needed to bring available stock up to the identified starting optimal quantity for the period.

1. Continuous Uniform Distribution

In this situation, let g(u) denote the continuous uniform demand distribution function whose characteristics have been previously described. Then, equation (2) may be directly used as follows in evaluating the optimal inventory quantity:

$$f(x;g) = \min_{y \geq x}\{C[y-x] + L(y;g)\}$$

$$= \min_{y \geq x}\left\{C[y-x] + H\int_{0}^{y}(y-u)g(u)\,du + P\int_{y}^{\infty}(u-y)g(u)\,du\right\}.$$

Since the function g(u) is zero for values of $u > n$ from equation (4), the TVC function to be minimized may be rewritten as:

$$TVC_y = C[y-x] + \frac{H}{n}\int_{0}^{y}(y-u)\,du + \frac{P}{n}\int_{y}^{n}(u-y)\,du.$$

Carrying out the integration results in:

$$TVC_y = \frac{(H+P)y^2}{2n} + (C-P)y + \frac{nP}{2} - xC.$$

Since Karlin has proven that the general TVC function resulting from his model is convex [Ref. 3: 236], minimization of this continuous function may be accomplished by taking the derivative with respect to y, setting it equal to 0, and solving for the optimal y. Thus,

$$0 = \frac{d}{dy} = \frac{(H+P)y}{n} + (C-P)$$

or,

$$y^* = \frac{n(P-C)}{H+P} . \qquad (5)$$

It can be seen from equation (5) that if the shortage penalty, P, is less than or equal to the unit cost, C, the optimal course of action is not to stock the item at all. This result occurs in the $P \leq C$ situation because it is cheaper to incur the penalty cost for being out of stock than it is to make the ordering cost investment needed to bring the item into the inventory. It should also be observed that the previous period closing inventory balance, x, has no impact on the value of y*.

2. Discrete Uniform Distribution

As previously noted, discrete demand distributions require a slightly different approach for evaluating y* than used with continuous functions because the required demand

probabilities are only defined for integer u values. Accordingly, the method of taking finite differences must be applied. The technique of finite differences calls for determining the largest integer value of y such that the expression $(TVC_y - TVC_{y-1})$ is less than zero. In this manner, that initial stock quantity which generates the lowest expected TVC will be identified since the cost function is convex and it can be shown that non-integer y values can never be optimal.

Assume that g(u) represents the discrete uniform demand distribution. To use this particular distribution in a single-period model, the TVC function must be modified to accommodate the discrete case. This may be accomplished as follows:

$$TVC_y = C[y-x] + L(y;g)$$

$$= C[y-x] + H\sum_{u=0}^{y-1}(y-u)g(u) + P\sum_{u=y}^{\infty}(u-y)g(u).$$

Substitution of g(u) from equation (4) results in:

$$TVC_y = C[y-x] + \frac{H}{n}\sum_{u=0}^{y-1}(y-u) + \frac{P}{n}\sum_{u=y}^{n}(u-y).$$

By making the necessary substitutions and performing appropriate cancellations, it can be shown that:

$$TVC_y - TVC_{y-1} = \frac{(H+P)y + nC - (n+1)P}{n}.$$

Hence, y* is the largest y value such that the above formulation is negative. This condition corresponds to the largest value of y for which:

$$y < \frac{(n+1)P - nC}{H+P} \quad . \tag{6}$$

As with the continuous case, the stockout penalty charge must be greater than the unit price if y* is to take on positive values. Also, y* is again independent of x.

3. Binomial Distribution

The binomial distribution is a more complex example of a discrete demand function than its uniform counterpart since this distribution will take on different probability values for possible demand quantities within the production range. Again, the method of finite differences is appropriate for determining y*.

Accordingly, let g(u) represent the probability that demand during the period in question will be u units, given a production schedule, n, and a repair part replacement factor, p. Then, as in the discrete uniform case,

$$TVC_y = C[y-x] + H\sum_{u=0}^{y-1}(y-u)g(u) + P\sum_{u=y}^{n}(u-y)g(u)$$

$$= C[y-x] + Hy\sum_{u=0}^{y-1}g(u) - H\sum_{u=0}^{y-1}ug(u) + P\sum_{u=y}^{n}ug(u) - Py\sum_{u=y}^{n}g(u).$$

By combining terms and recognizing that the mean of the binomial distribution is equal to np, this may be simplified to:

$$TVC_y = C[y-x] + (H+P)y\sum_{u=0}^{y-1} g(u) - (H+P)\sum_{u=0}^{y-1} ug(u) + P(np-y).$$

Using this result, it can be shown that:

$$TVC_y - TVC_{y-1} = (H+P)\sum_{u=0}^{y-1} g(u) + (C-P).$$

As with the discrete uniform case, y* is the largest y value for which the previous expression is negative. This condition corresponds to the largest value of y for which:

$$\sum_{u=0}^{y-1} g(u) < \frac{P-C}{H+P} . \qquad (7)$$

The same conclusions about the relationship between P and C as well as between y* and x can be drawn from this model as were identified in the two previous derivations.

G. SINGLE-PERIOD EXAMPLE

The following example is provided to illustrate the single-period models. Assume that the following data is available:

n = 10 units, C = \$100, H = \$50, P = \$200, x = 4 units.

Using the continuous uniform density function to represent the demand distribution, the optimal opening inventory balance may be computed from equation (5). Thus,

$$y^* = \frac{10(200-100)}{50+200} = 4 \text{ units.}$$

Since this quantity is equal to x, no order for additional stock should be placed.

Similarly, if g(u) is assumed to be the discrete uniform demand distribution, equation (6) shows that y* will be the largest integer y value such that:

$$y < \frac{(11)(200) - (10)(100)}{50 + 200} = 4.80.$$

Therefore, y* = 4 units. Again, the optimal procedure is not to order additional stock since a quantity of 4 units is already available.

Finally, in the binomial model with the replacement rate, p, assumed to be 0.5, equation (3) can be applied to generate the appropriate demand probability distribution values. This information can then be used in equation (7) to evaluate y* as 5 units. In this case, one additional unit should be ordered to bring the current inventory balance up to y*.

H. SUMMARY OF KEY FINDINGS

The major results derived in this chapter as applicable to single-period inventory models may be summarized as follows:

1. The optimization process identifies a critical number which is compared to the previous period closing balance in order to determine whether stock replenishment action is appropriate, and, if so, for what quantity.

2. y* values may be obtained for the demand distributions examined by applying equations (5), (6), and (7).

3. The shortage penalty charge, p, must exceed the unit price, C, for the models to produce optimal initial inventory values which are positive.

4. y^* is independent of the previous period closing balance, x.

III. DUAL-PERIOD MODELS--UNIFORM DISTRIBUTION

A. DISCUSSION

The single-period model developed in Chapter II will now be extended in an attempt to take advantage of production information for an additional period. However, the requirement to project costs for the second period is dependent on the actual demand during the first period, thus making the two-period model considerably more complex than the single-period situation. The approach taken in this chapter will be to first describe the general form of the dual-period model as well as its multi-period counterpart. Then, the specific model associated with the continuous uniform demand distribution function will be derived since this model permits explicit evaluation of the optimal starting inventory quantity. Finally, the effects of changing the demand distribution to the discrete uniform probability mass function will be explored. This will establish a degree of familiarity in dealing with discrete demand functions and will lay the groundwork for the dual-period binomial model which is examined in Chapter IV. In all two-period system derivations, the general assumptions made in Section B of Chapter II will continue to apply.

B. OBJECTIVE FUNCTION--DUAL/MULTI-PERIOD MODELS

As accomplished for the single-period case, the Karlin model can be adapted to provide an inventory system which determines optimal stock quantities based on projected production data for several quarters. This multi-period model assumes that demand is represented by a sequence of independent random variables which cover successive periods and are not necessarily identically distributed [Ref. 3: 233]. One familiar characteristic of such a model is that the resulting optimal inventory quantity for the first period is a critical number which has the same properties as did y* in the single-period system. An additional feature of the two-period model is that evaluation of the TVC function for any opening inventory value, y, is dependent on the computation of several optimal results from the second period considered alone. This will make determination of the optimal two-period inventory quantity a much more difficult task than was the case for the one-quarter model.

1. Dual-Period

Assume that x and y represent the previous period closing inventory balance and the present quarter opening quantity, respectively. Additionally, let g and h be continuous demand probability distributions for the next two periods. Then, the minimum expected TVC for a two-period model will be denoted f(x;g,h) where [Ref. 3: 235],

$$f(x;g,h) = \min_{y \geq x}\{C[y-x] + L(y;g) + f(0;h)\int_{y}^{\infty} g(u)\,du + \int_{0}^{y} f(y-u;h)\,g(u)\,du\}. \quad (8)$$

As with the single-period case, that y value which generates f(x;g,h) will be called y*.

In reviewing equation (8), it should be noted that the first two terms correspond, respectively, to procurement and surplus/shortage costs during the first period for the applicable opening inventory value. The third term represents the expected optimal total costs incurred in the second period as a result of having no initial stock balance in that period. Finally, the last term is the expected total optimal costs for the second quarter, given a positive starting inventory balance in that period as a consequence of the probability that demand during the first period is less than the initial stock quantity.

2. Multi-Period

Although multi-period inventory systems beyond the dual-period case are not within the scope of this thesis, the general finite-period model as adapted from Karlin should also be mentioned [Ref. 3: 235]. Accordingly, assume that $g_1, g_2, \ldots, g_n$ represent the demand probability distributions for the next n quarters. As with the preceding systems, the minimum TVC for the n-period model will be denoted $f(x;g_1,g_2,\ldots,g_n)$ and evaluated as:

$$f(x;g_1,g_2,\ldots,g_n) = \min_{y\geq x}\{C[y-x] + L(y;g_1) + f(0;g_2,g_3,\ldots,g_n)\int_y^\infty g_1(u)\,du + \int_0^y f(y-u;g_2,g_3,\ldots,g_n)\,g_1(u)\,du\}.$$

As can be seen, the individual terms in this formulation closely correspond to the previous description of a two-period model.

C. GENERAL DUAL-PERIOD RESULTS

Before proceeding to examine specific dual-period models, it is appropriate to summarize several important theorems generated by Karlin. The results which follow are dependent on the concept of stochastic ordering which is defined in the next paragraph.

Assume that $(g_1,g_2,\ldots,g_n)$ and $(h_1,h_2,\ldots,h_n)$ represent two sequences of continuous demand distributions for periods 1 through n. Then, the sequence g_i is said to be <u>stochastically smaller</u> than the sequence h_i if:

$$\int_0^y g_i(u)\,du \geq \int_0^y h_i(u)\,du$$

for i = 1,2,...,n and all non-negative y values. In other words, demands based on the function g_i have a larger probability of assuming smaller values than those generated by h_i. A simple single-period example of stochastic ordering can be

seen by comparing two continuous uniform distributions based on different production schedules. In this situation, the distribution resulting from the lower expected workload is stochastically smaller than the one associated with the higher production schedule.

Using this concept, several key results as proven by Karlin and then adapted to a dual-period model are listed as follows:

1. If g and h are continuous demand distributions for the next two periods and g is stochastically smaller than h, then the starting optimal inventory quantity for the two-period model generated by the demand sequence (g,h) is greater than or equal to the single-period y* value determined solely by g.

2. If g and h are continuous demand distributions for the next two periods and g is stochastically smaller than h, then the optimal inventory quantity for the first period based on the sequence (g,h) is equal to the y* value identified by the sequence (g,g). This is a particularly useful result for verifying information provided by specific models derived in the remainder of this thesis.

D. CONTINUOUS UNIFORM MODEL

The first specific dual-period model to be evaluated will be that system which is associated with the continuous uniform demand probability distribution. Accordingly, let m and n be the production schedules for a particular component in two successive rework periods, and g and h be the corresponding continuous uniform demand distributions. By applying equation (8), the optimal result for the two-period inventory model can be expressed as:

$$f(x;g,h) = \min_{y \geq x}\{C[y-x] + L(y;g) + f(0;h)\int_{y}^{\infty} g(u)\,du + \int_{0}^{y} f(y-u;h)\,g(u)\,du\}.$$

As will shortly be demonstrated, evaluation of the first three terms in this cost function is a relatively straightforward process. However, two separate cases must be considered in assessing the last term.

1. Common Objective Function Terms

The individual terms in the cost equation may be simplified as follows:

a. $L(y;g) = H\int_{0}^{y} (y-u)\,g(u)\,du + P\int_{y}^{m} (u-y)\,g(u)\,du.$

Applying the appropriate value of g and carrying out the integration results in:

$$L(y;g) = \frac{(H+P)y^2}{2m} - Py + \frac{mP}{2}.$$

b. $\int_{y}^{\infty} g(u)\,du = \int_{y}^{m} g(u)\,du = \frac{m - y}{m}.$

c. The previous chapter demonstrated the existence of a critical number associated with any single-period continuous uniform model (see equation (5)). This number determines whether or not a stock reorder should be submitted as well as the appropriate quantity in those situations requiring reorder action. Accordingly, let k denote the critical number

pertinent to just the second quarter of this two-period model. Based on previous results, a first period ending balance of zero implies that a requisition for k units must be submitted to obtain the optimal inventory balance at the start of the second period. Thus,

$$f(0;h) = kC + L(k;h)$$
$$= kC + H\int_0^k (k-u)h(u)\,du + P\int_k^\infty (u-k)h(u)\,du.$$

When this is evaluated, it reduces to:

$$f(0;h) = \frac{(H+P)k^2}{2n} + (C-P)k + \frac{nP}{2}.$$

d. Now, in the general case, the properties of k may be applied to show that:

$$f(y-u;h) = \begin{cases} L(y-u;h) & \text{if } y-u \geq k \\ C[k-(y-u)] + L(k;h) & \text{if } y-u < k. \end{cases}$$

Therefore, it may be concluded that:

$$\int_0^y f(y-u;h)g(u)\,du = \int_0^{y-k} L(y-u;h)g(u)\,du + \int_{y-k}^{y} \{C[k-(y-u)] + L(k;h)\}g(u)\,du. \quad (9)$$

In the case where $y < k$, the first term drops out while the lower bound over which the integration is performed on the second term becomes zero. Thus, two different cases need to be evaluated.

2. Case 1

In this case, $y \geq k$, so:

$$\int_0^{y-k} L(y-u;h)\,g(u)\,du = \frac{1}{m}\int_0^{y-k} \left\{\frac{(H+P)}{2n}(y-u)^2 - P(y-u) + \frac{nP}{2}\right\}du.$$

Also,

$$\int_{y-k}^{y} \{C[k-(y-u)] + L(k;h)\}g(u)\,du = \frac{1}{m}\int_{y-k}^{y} C[k-(y-u)]\,du + \frac{1}{m}\int_{y-k}^{y} \left\{\frac{(H+P)k^2}{2n} - Pk + \frac{nP}{2}\right\}du.$$

After evaluating these integrals, the entire objective function for case 1 may be constructed by combining this result with the formulas for the common objective function terms from subsection 1 above. Finally, after taking the first derivative with respect to y and setting it equal to 0, the following result is obtained:

$$y^* = \frac{-nH + \sqrt{n^2H^2 - n[P^2(n-2m) + nC^2 + 2CP(m-n) + 2mH(C-P)]}}{H+P}. \qquad (10)$$

3. Case 2

Since $y < k$ in this case, the last term in the objective function given by equation (9) is now:

$$\int_0^{y} f(y-u;h)\,g(u)\,du = \int_0^{y} \{C[k-(y-u)] + L(k,h)\}g(u)\,du$$

$$= \frac{1}{m}\int_0^{y} C[k-(y-u)]\,du + \frac{1}{m}\int_0^{y} \left\{\frac{(H+P)k^2}{2n} - Pk + nP\right\}du.$$

After carrying out the integration, the entire cost function for case 2 is formed as before by adding this result to the common objective function terms. Using the calculus, the optimal y value can then be identified as:

$$y^* = \frac{m(P-C)}{H+P-C} \quad . \tag{11}$$

It should be noted that, in both cases, the formula for computing y* is independent of the closing stock balance, x, from the previous period. This result corresponds to an identical finding for the single-period continuous uniform model. Also, the value of P must again be greater than C for positive y* values to occur.

4. Breakpoint Between Cases 1 and 2

Prior to examining the results obtained from this model for specific parameter values, it is useful to note that the two distinct cases are equivalent at y = k. Therefore, the breakpoint between these alternatives may be calculated in terms of parameter values by setting the optimal result from either case equal to the expression for k as derived from equation (5). In particular, using the case 2 result,

$$y^* = \frac{m(P-C)}{H+P-C} = \frac{n(P-C)}{H+P} = k.$$

Therefore,

$$P = \frac{nH - nC - mH}{m-n} \quad . \tag{12}$$

For P values greater than the right-hand side of this expression, equation (11) should be used to calculate the optimal inventory quantity, while smaller P values require the use of equation (10). Equation (12) may also be utilized to obtain a definition of this breakpoint between the cases in terms of the other parameters.

E. EXAMPLE OF THE CONTINUOUS UNIFORM MODEL

Assume that the projected production schedule for the next two quarters is 100 and 200 units, respectively. Additionally, let C = \$30, H = \$20 and x = 0. Using the previous findings, a comparison of the optimal stock quantities at the beginning of the next period for the single and dual-period models may be conducted for various values of P. The single-period optimization utilizes the projected first period workload but ignores subsequent production information. Curves of the y* values obtained from both models are displayed in Figure 3.1. In the dual-period model, a P value of 40 was computed as the breakpoint between cases 1 and 2 by using equation (12).

These results are intuitively appealing for several reasons. First, increasing the stockout cost causes the optimal initial inventory to take on larger and larger values under both systems. Additionally, the dual-period model, which considers projected demand beyond the first period, computes constantly larger inventory quantities than the single-period

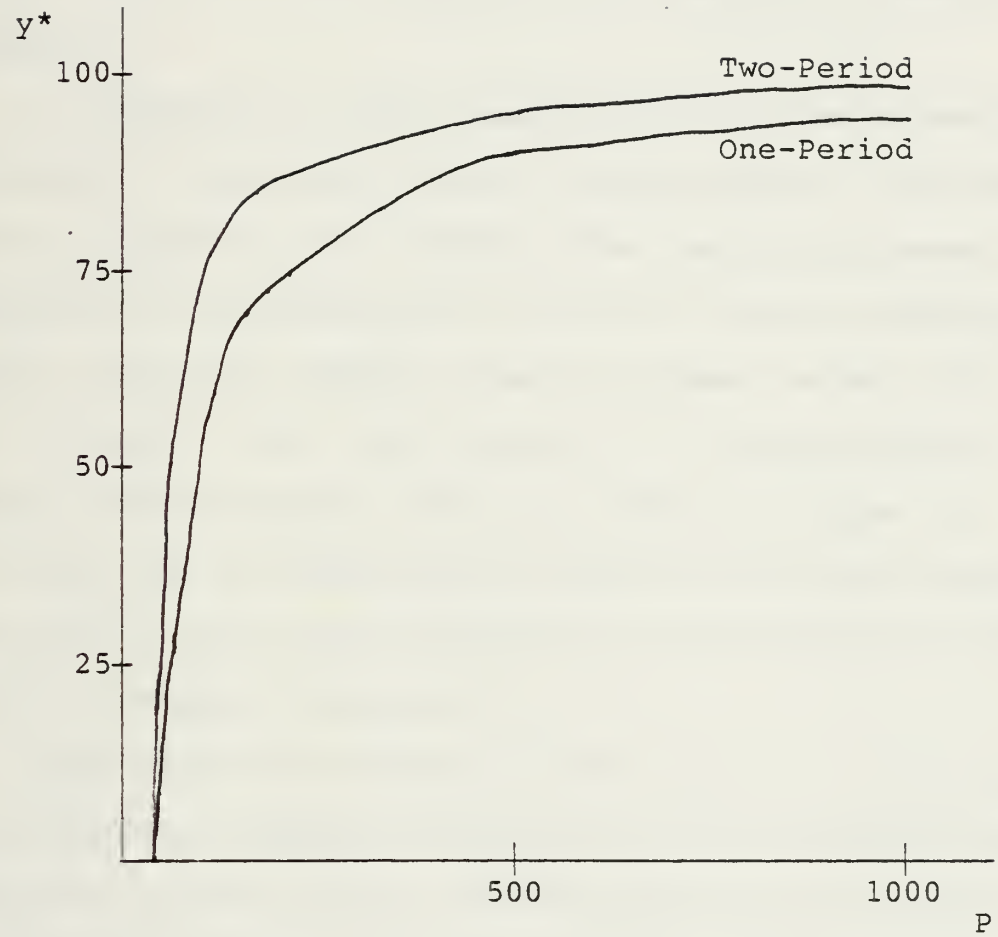

Figure 3.1 Continuous Uniform Model: One vs Two-Period

model. Since the demand distributions are stochastically increasing, this is in agreement with the findings of Karlin which were previously discussed. Finally, extremely large stockout penalty costs generate nearly equivalent results under both systems as the initial inventory is maintained at a high level to protect against this exorbitant penalty charge.

An assumption in this example was that the production schedule in the second quarter exceeded that of the first period. Suppose that, instead, the rework requirements are projected to be decreasing over time. Let the parameters used in the first example remain the same, except that now $m = 200$ and $n = 100$. The curves of y^* corresponding to the single and dual-period models are shown in Figure 3.2. The y^* values for the dual-period system are derived entirely from case 2 since the application of the breakpoint formula yields a negative value of P.

These values are similar to the results of the increasing production schedule situation as the two-period system generates higher optimal inventory quantities than the one-quarter model despite the projected decrease in rework activity. This leads to the general conclusion that a two-period formulation can be applied in both increasing and decreasing production schedule situations to identify a y^* solution yielding lower TVC results than those derived from a single-period y^* value.

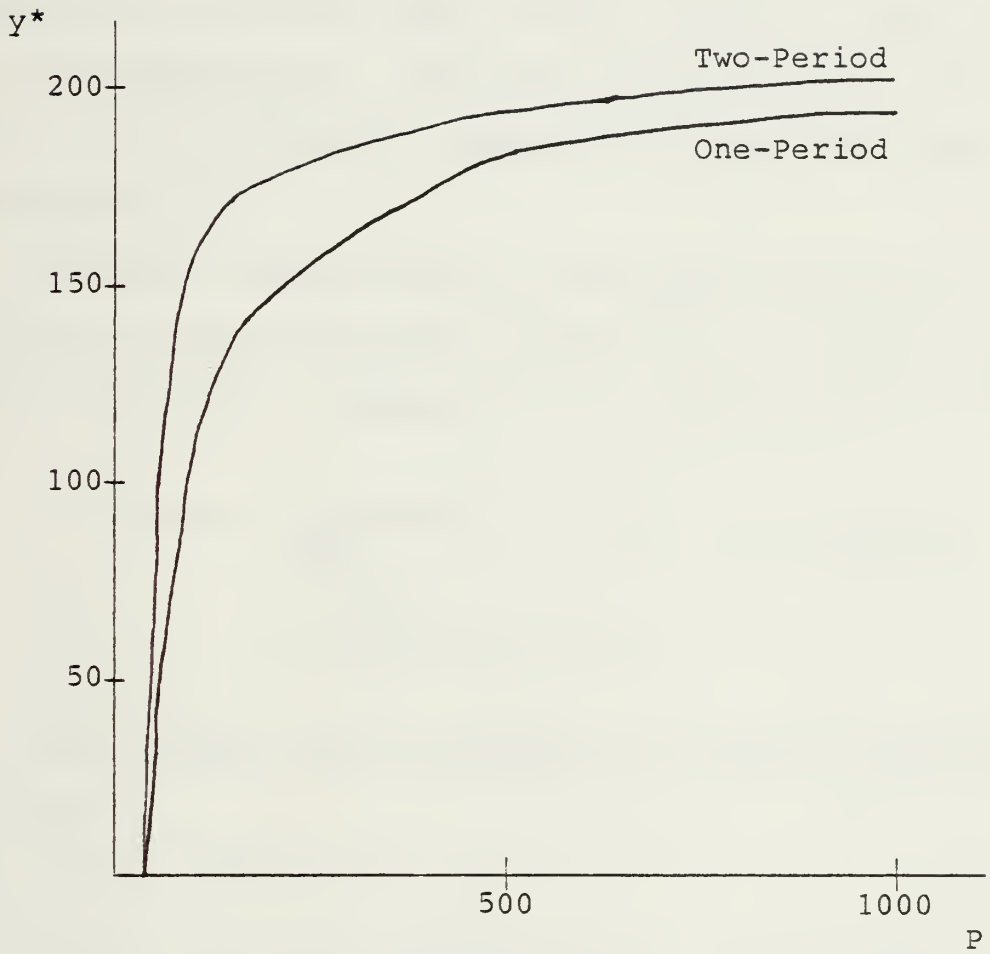

Figure 3.2 Continuous Uniform Model: Decreasing Production

F. DISCRETE UNIFORM MODEL

A logical extension of the continuous uniform model is to consider the discrete case in which both demand and inventory quantities may take on only integer values. This formulation provides a closer representation of the actual situation at the NARF where integer values of these parameters are appropriate. Since the demand function under this model is not continuous, finite differences are needed in determining y*.

As before, assume that the production schedules for the next two periods are m and n respectively. Equation (8) can then be modified to accommodate the discrete case as follows:

$$f(x;g,h) = \min_{y \geq x} \left\{ C[y-x] + L(y;g) + f(0;h) \sum_{u=y}^{m} g(u) + \sum_{u=0}^{y-1} f(y-u;h)\, g(u) \right\}.$$

Now, $TVC_y - TVC_{y-1}$ may be examined on a term by term basis as follows:

1. $C[y-x] - C[(y-1) - x] = C.$

2. $$L(y;g) - L(y-1;g) = H \sum_{u=0}^{y-1} (y-u)g(u) + P \sum_{u=y}^{m} (u-y)g(u) - H \sum_{u=0}^{y-2} [(y-1)-u]g(u) - P \sum_{u=y-1}^{m} [u-(y-1)]g(u).$$

After several steps, this may be reduced to:

$$L(y;g) - L(y-1;g) = \frac{(H+P)y - (m+1)P}{m}.$$

3. As before,

$$f(0;h) = kC + L(k;h)$$

$$= kC + \frac{H}{n} \sum_{u=0}^{k-1} (k-u) + \frac{P}{n} \sum_{u=k}^{n} (u-k)$$

which is independent of y. The finite differences result for the third term is therefore

$$= -\left\{ \frac{kC}{m} + \frac{H}{mn} \sum_{u=0}^{k-1} (k-u) + \frac{P}{mn} \sum_{u=k}^{n} (u-k) \right\}.$$

4. The last term of the cost equation must again be considered in two separate cases. For case 1,

$$\sum_{u=0}^{y-1} f(y-u;h)g(u) = \sum_{u=0}^{y-k} L(y-u;h)g(u) + \sum_{u=y-k+1}^{y-1} C[k-(y-u)]g(u)$$

$$+ \sum_{u=y-k+1}^{y-1} L(k;h)g(u).$$

However, the application of the method of finite differences in this case fails to provide the necessary cancellations which simplify the evaluation process and, thus, justify use of this technique. As a result, a complete enumeration of the TVC results for all possible y values offers a better approach to solving case 1 than does finite differences. Since this procedure is employed in Chapter IV to evaluate the dual-period binomial model, it will not be duplicated here. Rather, the results for case 2 under finite differences will be shown.

Case 2

Since $y < k$,

$$\sum_{u=0}^{y-1} f(y-u;h)g(u) = \sum_{u=0}^{y-1} C[k-(y-u)]g(u) + \sum_{u=0}^{y-1} L(k;h)g(u).$$

Taking finite differences results in:

$$= \frac{C(k-y)}{m} + \frac{H}{mn} \sum_{u=0}^{k-1} (k-u) + \frac{P}{mn} \sum_{u=k}^{n} (u-k).$$

Putting the previous results all together, the following expression for the difference in variable costs between quantities of stock y and (y-1) may be obtained for case 2 of the two-period discrete uniform model:

$$TVC_y - TVC_{y-1} = \frac{(H+P-C)y + mC - (m+1)P}{m}.$$

Thus, the optimal y is the largest integer quantity for which the above expression is negative. This condition corresponds to the largest value of y for which:

$$y < \frac{(m+1)P - mC}{H+P-C}. \qquad (13)$$

If there is also an integer value of y which is equal to the right-hand side of equation (13), then it is an alternate optimal solution. As in the previous model, the optimal y value is independent of the previous period closing balance, x, and P must be greater than C for y* to be positive.

G. EXAMPLE OF THE DISCRETE UNIFORM MODEL

Figure 3.3 presents the results obtained for a range of stockout costs utilizing the discrete uniform demand distribution in single and dual-period models as derived above and in Chapter II. The values for m, n, x, C and H remain the same as in Figure 3.1. Only relatively large values of P are analyzed to ensure that the breakpoint between cases 1 and 2 is exceeded. The y* values obtained are consistent with those computed from the continuous uniform approach.

H. SUMMARY OF KEY FINDINGS

The principal results obtained in this chapter for two-period inventory models based on uniform demand distributions are summarized as follows:

1. As in the single-period case, a critical number which determines the necessity of ordering additional stock at the beginning of the first period as well as the optimal order quantity can be identified. This number is independent of the previous period closing balance, and may be computed for the various cases described by applying equations (10), (11) and (13). Additionally, the shortage cost rate must be greater than the unit price if the item is to be stocked at all.

2. The critical number for the first quarter of a dual-period model is always greater than or equal to the optimal result for the corresponding one-period model.

3. In both increasing and decreasing production schedule situations, the two-period model identifies a y* quantity which yields lower total costs than those generated by a one-period y* value.

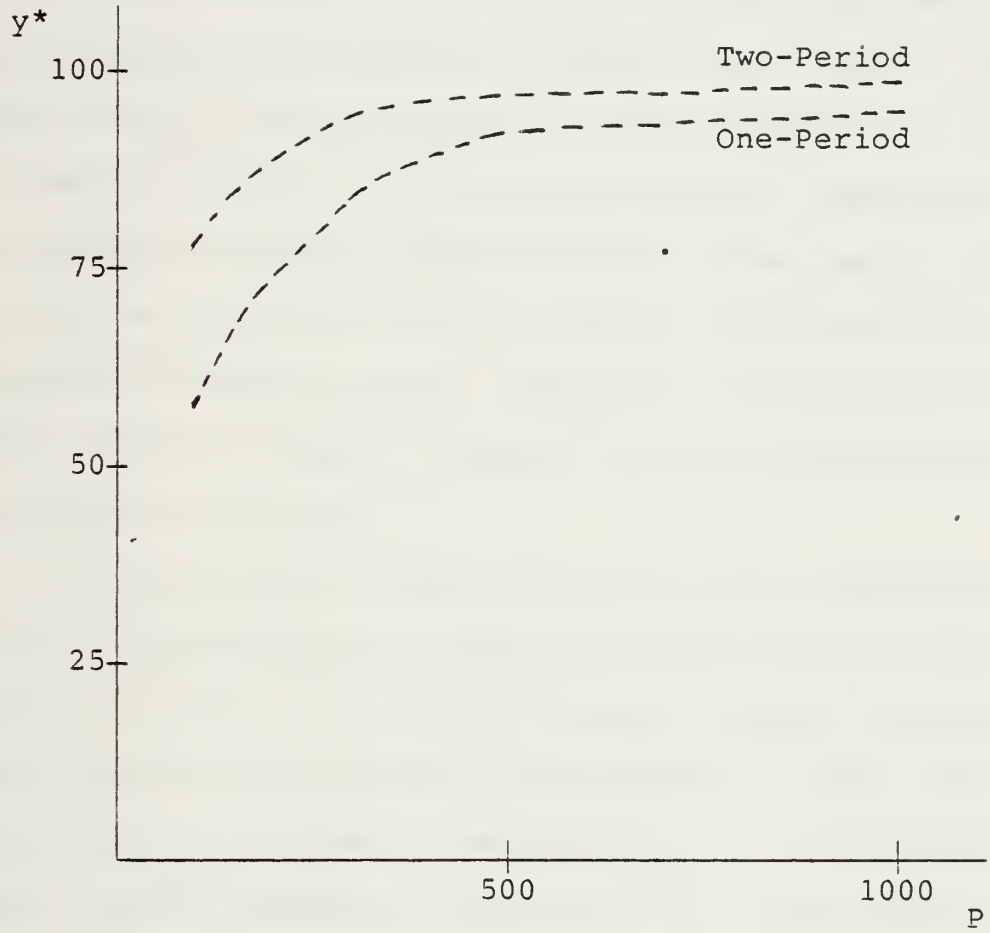

Figure 3.3 Discrete Uniform Model: One vs Two-Period

IV. DUAL-PERIOD BINOMIAL MODEL

A. DISCUSSION

Now that an understanding of the general rationale behind a two-period inventory system has been developed using the uniform demand distribution, attention will turn to the binomial density function which is more pertinent to a NARF. As with the discrete uniform case, explicit computation of the optimal inventory quantity for this dual-period model will not be possible since the demand distribution is only defined at integer values. Therefore, a technique of comparing the TVC values for possible initial inventory quantities must be employed.

The computation of total costs for this distribution will be more complex than for the uniform distribution since the density function will take on different values for various demand possibilities within the production range. The same complication will also be encountered as in the previous examples whereby separate evaluation of the last term of the objective function for two distinct cases must be accomplished. As a consequence, the technique of finite differences will not provide any advantages over using the TVC values directly. Accordingly, the general approach to this particular model will be to first derive the entire TVC formulation for both cases, and then to use these formulas to

evaluate the expected variable costs incurred for each possible starting inventory value. From this enumeration, the optimal y associated with the minimum TVC can be selected.

B. THE MODEL

Let g and h represent the binomial demand distributions for the next two periods derived from an historical replacement factor, p, and production schedules m and n. As with the discrete uniform model, equation (8) must be modified to yield an expression for TVC associated with the opening inventory balance, y, and a previous period ending quantity, x. The y* value has been shown to be independent of the previous period closing balance for all models previously evaluated in this thesis, and a similar rationale can be employed in this particular situation. Therefore, x will be assumed to be zero for convenience in formulating the model. Then,

$$TVC_y = Cy + L(y;g) + f(0;h) \sum_{u=y}^{m} g(u) + \sum_{u=0}^{y-1} f(y-u;h)g(u).$$

1. Common Objective Function Terms

The evaluation of the first three terms of the objective function will be accomplished independently of the last term which involves separate cases contingent on the value of the critical number for the second period. Thus,

a. The first term, Cy, and one component of the third term, g(u), cannot be simplified.

b. $$L(y;g) = H\sum_{u=0}^{y-1}(y-u)g(u) + P\sum_{u=y}^{m}(u-y)g(u)$$

$$= Hy\sum_{u=0}^{y-1}g(u) - H\sum_{u=0}^{y-1}ug(u) + P\sum_{u=y}^{m}ug(u) - Py\sum_{u=y}^{m}g(u).$$

However, the following substitution may be made:

$$\sum_{u=y}^{m}g(u) = \sum_{u=0}^{m}g(u) - \sum_{u=0}^{y-1}g(u) = 1 - \sum_{u=0}^{y-1}g(u).$$

Also,

$$\sum_{u=y}^{m}ug(u) = \sum_{u=0}^{m}ug(u) - \sum_{u=0}^{y-1}ug(u).$$

Now, the first term on the right-hand side of the expression immediately above is the mean of g(u). Hence,

$$\sum_{u=y}^{m}ug(u) = pm - \sum_{u=0}^{y-1}ug(u).$$

With these substitutions, L(y;g) reduces to:

$$L(y;g) = (H+P)y\sum_{u=0}^{y-1}g(u) - (H+P)\sum_{u=0}^{y-1}ug(u) + pmP - Py. \qquad (14)$$

c. As in previous formulations, let k denote the critical number derived from an independent evaluation of the second period. This may be accomplished by using equation (7) as developed in Chapter II. Then, using the variable s to represent demand in the second period,

$$f(0;h) = kC + L(k;h)$$

$$= kC + H\sum_{s=0}^{k-1}(k-s)h(s) + P\sum_{s=k}^{n}(s-k)h(s).$$

Applying the same simplifications as were used to obtain equation (14) results in:

$$f(0;h) = (C-P)k + (H+P)k \sum_{s=0}^{k-1} h(s) - (H+P) \sum_{s=0}^{k-1} sh(s) + pnP.$$

2. Case 1

For the case where $y \geq k$, we recall from Chapter III that:

$$\sum_{u=0}^{y-1} f(y-u;h)g(u) = \sum_{u=0}^{y-k} L(y-u;h)g(u) + \sum_{u=y-k+1}^{y-1} C[k-(y-u)]g(u) + \sum_{u=y-k+1}^{y-1} L(k;h)g(u).$$

Now, this expression can be simplified using the definition of $L(y;h)$ as contained in equation (1), the properties of the binomial distribution, and the variable s to represent demand in the second period. After simplification, the entire objective function for this case becomes:

$$\begin{aligned} TVC_y = {} & (C-P)y + (H+P-C)y \sum_{u=0}^{y-1} g(u) - (H+P-C) \sum_{u=0}^{y-1} ug(u) \\ & + pmP + Cy \sum_{u=0}^{y-k} g(u) - C \sum_{u=0}^{y-k} ug(u) \\ & + \sum_{u=y-k+1}^{m} \left\{ (H+P)k \sum_{s=0}^{k-1} h(s) - (H+P) \sum_{s=0}^{k-1} sh(s) + pnP + k(C-P) \right\} g(u) \\ & + \sum_{u=0}^{y-k} \left\{ (H+P)(y-u) \sum_{s=0}^{y-u-1} h(s) - (H+P) \sum_{s=0}^{y-u-1} sh(s) + pnP - P(y-u) \right\} g(u). \end{aligned} \tag{15}$$

3. Case 2

In this situation with $y < k$,

$$\sum_{u=0}^{y-1} f(y-u;h)g(u) = \sum_{u=0}^{y-1} C[k-(y-u)]g(u) + \sum_{u=0}^{y-1} L(k;h)g(u).$$

Using an identical approach to that of case 1, an evaluation of this expression may be performed. After retrieving all terms in the objective function, making the appropriate combinations, and noting that the summation from 0 to m of g(u) equals 1, the result for case 2 is:

$$TVC_y = (C-P)y + (H+P-C)y \sum_{u=0}^{y-1} g(u) - (H+P-C) \sum_{u=0}^{y-1} ug(u) + pmP$$

$$+ (H+P)k \sum_{s=0}^{k-1} h(s) - (H+P) \sum_{s=0}^{k-1} sh(s) + pnP + (C-P)k. \quad (16)$$

4. Conclusion

The critical number y* for a two-period binomial model may thus be identified by using equations (15) and (16) to compute the TVC values for all possible initial inventory quantities; it is that value of y which results in the minimum TVC. It should be noted that, if the assumption of x equaling zero is dropped, then the TVC equation for each y value will include a Cx term. Therefore, the relationship between TVC values associated with all y quantities will be unchanged by the addition of this common term.

C. EXAMPLES OF THE BINOMIAL MODEL

Because the objective functions developed in the two-period binomial model are rather complex, a computer program was written which accepts various system parameters and determines the optimal y value for the beginning of the first period. Unlike the continuous uniform model, identification of the breakpoint between cases 1 and 2 is not required as the program first determines k and then uses this value in calculating appropriate variable costs.

Results obtained from using this program for various values of P with

$$m = 10, \quad n = 20, \quad p = 0.5, \quad x = 0, \quad H = \$20, \quad C = \$30$$

are displayed in Figure 4.1. As in all previous single and dual-period models, values of P less than C imply that the item should not be stocked at all. Consequently, only P values above $30 are included in this figure.

These results are consistent with those obtained using the uniform distribution models in that the optimal value of y at the beginning of the first period increases with the stockout penalty. Additionally, the extra information contained in the two-period model causes its optimal result to be greater than or equal to that obtained from the single-period analysis. However, the difference in the y* values is never more than one. Thus, an examination of the benefits derived from the additional stock quantities in the dual-period model needs to be conducted.

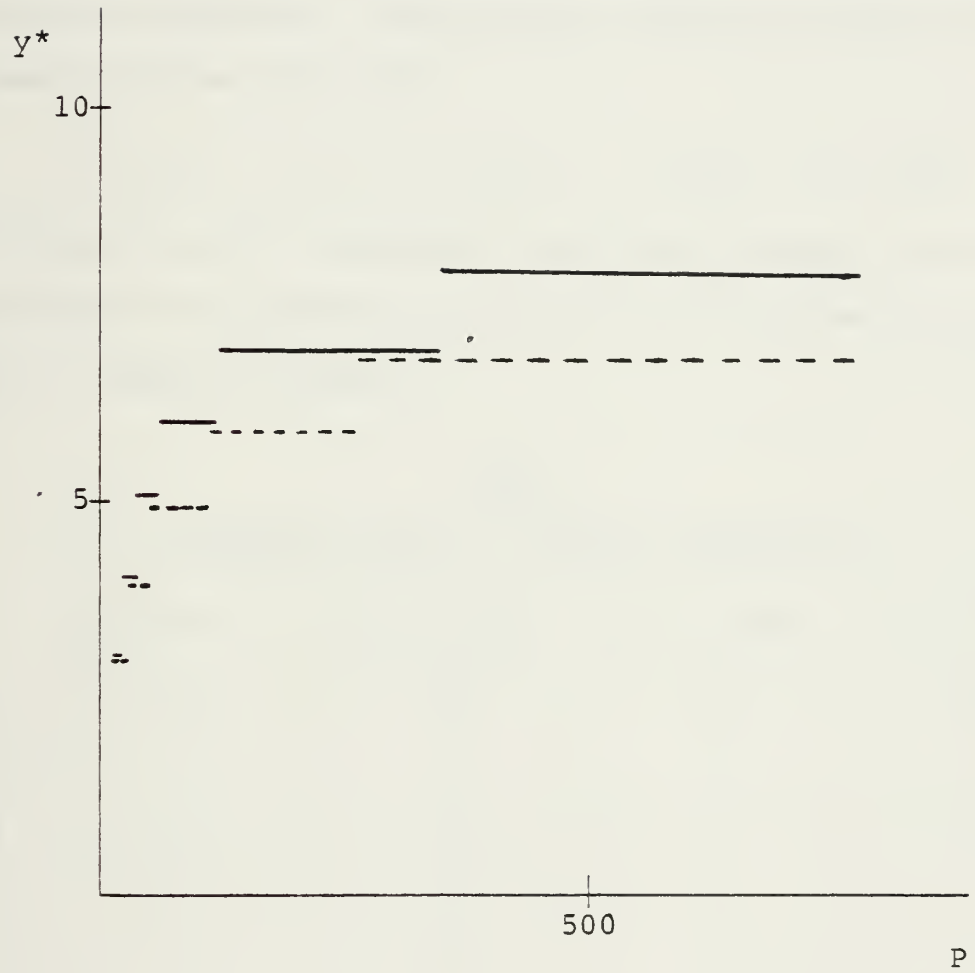

Figure 4.1 Binomial Model: One vs Two-Period

However, before performing this detailed analysis, it is useful to examine a decreasing production schedule situation as well as verify Karlin's result that a two-period inventory model based on stochastically increasing demand distributions generates y* values which are equal to the y* quantities produced by a dual-period inventory system based on a constant production schedule. Accordingly, assume that the following parameter values are given:

$$m = 10, \quad x = 0, \quad H = \$20, \quad C = \$30, \quad P = \$150$$

The optimal initial inventory quantities derived for various replacement rate values and second-period production schedules are shown in Table I.

TABLE I

Binomial Model: Varying Production

p	y* (n = 5)	y* (n = 10)	y* (n = 20)	y* (n ≥ 50)
0.9	10	10	10	10
0.8	9	9	9	9
0.7	8	9	9	9
0.6	8	8	8	8
0.5	6	7	7	7
0.4	5	6	6	6
0.3	4	5	5	5
0.2	3	3	3	3
0.1	2	2	2	2

Several conclusions may be drawn from Table I. First, the situation where the production schedule in the second

period is greater than or equal to ten fulfills the conditions for a stochastic demand increase, and, as noted by Karlin, this environment always produces a y* value equal to that of the constant production schedule situation. Additionally, the case where demand is stochastically decreasing, as depicted in the table under n = 5, identifies y* values which are nearly equivalent to the increasing production schedule situation. This may be interpreted to mean that the mere prospect of any demand in the second period is more important than the actual forecast quantity.

D. SUMMARY OF KEY FINDINGS

The major results of this chapter for the two-period binomial inventory model are summarized as follows:

1. The optimal initial inventory value is again a critical number which may be computed by identifying that y value which generates the minimum TVC in equations (15) and (16). As in the previous single and dual-period models, y* is independent of the previous period closing balance, x, and will only take on positive values if the shortage cost rate for an item exceeds its unit price.

2. An increasing production schedule will produce a y* value identical to the situation in which the production workload is constant.

V. TWO-PERIOD BINOMIAL MODEL SENSITIVITY ANALYSIS

A. DISCUSSION

In this chapter, a detailed analysis of the two-period binomial model is conducted. The first portion of this analysis is a parametric evaluation of the sensitivity of model results to changes in the unit price and the surplus and shortage cost rates. Then, a comparison of expected optimal TVC values generated by several alternative binomial models using a two-period time horizon is conducted. These alternative costs are arrived at by employing the standard two-period model, the two-period objective function using single-period y* values, and the single-period model for two distinct periods. Based on the results of this analysis, several general conclusions on the use of the two-period binomial model are drawn.

B. ORDERING, SURPLUS AND SHORTAGE COST SENSITIVITY ANALYSIS

One aspect of analyzing the dual-period binomial inventory model is to evaluate the sensitivity of the model to changes in various system parameters. Since the production schedules which generate the demand distributions are determined outside the scope of the model, the current emphasis will be on examining the effects of changes in the unit price and surplus/shortage cost parameters on system results.

Throughout this section, a constant production schedule of 10 units for each period is assumed, while the replacement factor is permitted to take on the values of 0.9, 0.5 and 0.1. Then, two of the three parameters of interest will be held constant so that the results obtained from varying the third may be examined.

1. Unit Price Changes

An assumption was made in Chapter II that the surplus cost rate for an item must be less than its unit price. Additionally, the preliminary analysis conducted on the binomial model in Chapter IV established that the shortage cost rate should be greater than the unit price for any particular part to be stocked. Thus, a range of potential values for C between H and P seems appropriate. Under the assumption that H = $250 and P = $1000, Table II displays optimal initial inventory quantities associated with the two-period model and the previously identified replacement factors for various values of C within the established range. The TVC calculated at y* and at values of y which differ from y* by one unit are also shown.

The general conclusion that may be drawn from this table is that y* is not particularly sensitive to changes in the unit price, except when the value of C approaches the ceiling level established by the shortage cost rate. However, when C does get close to P, the difference between expected variable costs for y* and the initial inventory

TABLE II

Binomial Model: Sensitivity Analysis of C

(1) p = 0.9

C	y*	TVC(y*-1)	TVC(y*)	TVC(y*+1)
250	10	5284.52	5185.84	---
500	10	9697.35	9685.86	---
750	9	14148.34	14030.29	14105.94
950	8	17322.59	17293.65	17322.81
990	7	17882.69	17876.01	17884.26

(2) p = 0.5

C	y*	TVC(y*-1)	TVC(y*)	TVC(y*+1)
250	6	3884.28	3757.35	3836.04
500	6	6230.46	6198.02	6322.36
750	5	8416.99	8355.68	8419.88
950	3	9818.16	9784.58	9786.48
990	2	9981.47	9974.27	9978.57

(3) p = 0.1

C	y*	TVC(y*-1)	TVC(y*)	TVC(y*+1)
250	1	2000.00	1284.53	1329.94
500	1	2000.00	1697.36	1895.92
750	1	2000.00	1990.08	2413.22
950	0	---	2000.00	2189.28
990	0	---	2000.00	2230.08

quantities immediately above and below y* is so small as to be insignificant. In fact, if a y value of 6 is used in the p = 0.5 model with C = $990, the expected TVC is only three percent higher than the corresponding TVC associated with the optimal y value of 2.

2. Surplus Cost Rate Changes

In this portion of the analysis, the range of potential surplus cost values will not be assumed to be bounded by other parameters, despite the assumption in Chapter II that

the surplus cost rate would be expected to be less than the unit price. This assumption about H is temporarily suspended so that the results obtained from a wide range of H values may be examined. Dual-period binomial model results for various H values were obtained for the three replacement factors when C = $500 and P = $1000, and these values are listed in Table III.

TABLE III

Binomial Model: Sensitivity Analysis of H

(1) p = 0.9

H	y*	TVC(y*-1)	TVC(y*)	TVC(y*+1)
25	10	9540.45	9382.39	---
100	10	9592.75	9483.54	---
250	10	9697.35	9685.86	---
500	9	10107.78	9871.69	10023.00
750	9	10216.14	10046.01	10360.17
1000	9	10296.70	10192.55	10669.53
1500	8	10741.10	10381.48	10409.28

(2) p = 0.5

H	y*	TVC(y*-1)	TVC(y*)	TVC(y*+1)
25	7	5780.84	5717.30	5723.55
100	7	5919.90	5918.99	5998.86
250	6	6230.46	6198.02	6322.36
500	5	6595.70	6472.72	6596.83
750	5	6714.84	6686.19	6967.13
1000	4	7076.17	6833.98	6899.66
1500	4	7228.52	7072.26	7326.60

(3) p = 0.1

H	y*	TVC(y*-1)	TVC(y*)	TVC(y*+1)
25	2	1540.45	1515.64	1760.36
100	1	2000.00	1592.75	1642.40
250	1	2000.00	1697.36	1895.92
500	1	2000.00	1856.89	2312.86
750	1	2000.00	1974.51	2712.50
1000	0	---	2000.00	2092.12
1500	0	---	2000.00	2327.35

The optimal initial inventory quantity decreases with increasing H but appears to be fairly insensitive to changes in the surplus cost rate. Additionally, the expected TVC at y* for H = $1500 are less than twenty-five percent higher than the results for H = $25 in all three cases, despite the fact that the surplus cost rate has been increased by a factor of sixty. These results imply that a precise estimate of the surplus cost rate is not required to obtain useful model results.

3. Shortage Cost Rate Changes

Finally, the last parameter to be considered in this section of the analysis is the shortage cost parameter. As was noted in an earlier chapter, this parameter must be greater than the applicable item's unit price if the part in question is to be stocked at all. Thus, the range of values for P will start at C and has no upper bound. Table IV depicts the dual-period model results obtained for various values of P when C = $500, H = $250 and the replacement factor takes on the three values as before.

The evidence from this table is that the optimal stock quantity at the beginning of the first period is insensitive to changes in the stockout cost rate when P is much larger than C. However, as the difference between the unit price of an item and its stockout cost grows small, the y* value changes more rapidly for relatively minor changes in P, especially in the case where p = 0.5. Additionally, the

TABLE IV

Binomial Model: Sensitivity Analysis of P

(1) p = 0.9

P	y*	TVC(y*-1)	TVC(y*)	TVC(y*+1)
550	8	9216.04	9187.09	9216.25
750	9	9627.15	9509.10	9584.75
1000	10	9697.35	9685.84	---
2000	10	10360.17	9999.98	---
5000	10	11406.20	9999.98	---
10000	10	13149.43	9999.99	---
20000	10	16636.25	9999.99	---

(2) p = 0.5

P	y*	TVC(y*-1)	TVC(y*)	TVC(y*+1)
550	3	5306.64	5273.04	5274.69
750	5	5857.42	5796.02	5859.15
1000	6	6230.46	6198.02	6322.36
2000	7	6953.12	6902.70	7061.15
5000	8	7664.06	7654.75	7858.92
10000	8	8328.12	8045.11	8192.78
20000	8	9298.81	8468.81	8507.14

(3) p = 0.1

P	y*	TVC(y*-1)	TVC(y*)	TVC(y*+1)
550	0	---	1100.00	1245.67
750	1	1500.00	1458.86	1784.82
1000	1	2000.00	1697.36	1895.92
2000	2	2394.71	2237.36	2631.49
5000	2	3851.30	2847.77	2897.38
10000	3	3695.53	3271.82	3624.67
20000	3	4719.71	3583.43	3751.84

difference in expected TVC between y* and initial inventory levels immediately above and below y* becomes much greater as P takes on larger values. For example, the optimal result of y = 8 for P = $20000 and p = 0.5 is nearly ten percent less expensive than the expected TVC at y = 7. The conclusion which may be drawn is that the NARF needs to have reasonably

accurate estimates of its shortage cost rates, particularly for high value items.

C. TVC COMPARISON--SINGLE VS DUAL-PERIOD MODELS

In this section, the difference in the expected TVC between single and dual-period binomial models is evaluated and analyzed. Comparison between the results obtained from the following distinct models is accomplished over a two-period cost horizon:

1. The first alternative involves the computation of expected total costs associated with y* as derived from the dual-period binomial model.

2. The second alternative computes an expected TVC value by a combined use of single and dual-period models. This is accomplished by substituting the y* value determined from a single-period model into the two-period objective function. This alternative, when contrasted with the pure dual-period result, depicts the differences between one and two-period models on a scale which allows for a meaningful comparison of the systems.

3. The third TVC result is determined by considering each quarter as a separate, single-period model. The y* values obtained for each period are then used in the single-period binomial objective function to calculate the aggregate expected TVC result. This alternative implies that any surplus at the end of the first quarter is disposed of and a buy for a quantity equivalent to the second-period y* is initiated.

Assume that the model parameters take on the following values:

$$C = \$500, \quad H = \$250, \quad P = \$1000, \quad p = 0.5, \quad x = 0.$$

Table V displays the TVC results for the three models. The numbers in parentheses immediately following the total costs for each alternative identify the y* value(s) used in that

TABLE V

Binomial Model: TVC Comparisons

Production Schedules	Alternative One--TVC	Alternative Two--TVC	Alternative Three--TVC
m=5, n=5	3316.41 (3)	3437.50 (2)	3546.87 (2,2)
m=5, n=10	4808.11 (3)	4933.11 (2)	5042.48 (2,5)
m=5, n=20	7625.41 (3)	7750.41 (2)	7859.79 (2,9)
m=10, n=5	4747.93 (5)	4747.93 (5)	5042.48 (5,2)
m=10, n=10	6198.02 (6)	6230.46 (5)	6538.08 (5,5)
m=10, n=20	9015.06 (6)	9047.78 (5)	9355.39 (5,9)

model in determining the expected TVC. The second of these numbers in parentheses for the third alternative is the k value for the second period which was described earlier in this thesis.

As anticipated, the optimal inventory value identified by the two-period binomial model produces a lower expected TVC than either of the other two alternatives. The more expensive result for the second alternative results from the optimal dual-period y* not being used, while the increased costs associated with the third alternative are a consequence of disposing of the first period excess and, thus, having no carryover inventory. Despite these differences, the cost variations between the various model results are relatively small. Table VI depicts the differences between alternatives one and two as well as between one and three as a percentage of the expected TVC calculated for alternative one.

TABLE VI

Binomial Model: Percentage Differences

Production Schedules	Alternatives One and Two	Alternatives One and Three
m=5, n=5	3.7	6.9
m=5, n=10	2.6	4.9
m=5, n=20	1.6	3.1
m=10, n=5	0.0	6.2
m=10, n=10	0.5	5.5
m=10, n=20	0.4	3.8

The most interesting result that can be seen from Table VI is that the percentage difference between alternative one and both other alternatives decreases as the production schedule for the second period increases, except in the case where the y* value for alternatives one and two is the same. In fact, by the time that the rework forecast for the first period reaches ten, the expected cost benefit to be derived from using the two-period y* instead of the single-period y* borders on the insignificant, thereby making the use of the more complex two-period model a questionable proposition.

One additional factor which needs to be considered before concluding this model comparison is the accuracy of production schedule information. The first quarter workload forecast is utilized in both single and dual-period models, and so an error in this parameter will have a similar impact on the results obtained from each system. On the other hand, since the second period projected rework schedule is only

considered in the two-period model, a variance in this parameter between the actual and forecasted workload will only affect the results obtained from the two-period model. Additionally, some difference between the actual and projected schedules is more likely to occur in the second period than in the first since current information is generally more accurate than that applying to a future period. While the impact of an inaccurate second period rework forecast will vary greatly from situation to situation, it will only serve to further reduce or even eliminate the limited advantages which the two-period model offers over its single-period counterpart.

D. SUMMARY OF KEY FINDINGS

The principal results identified in this chapter as applicable to the dual-period binomial model are listed as follows:

1. The model is very sensitive to the value of the shortage cost parameter and is relatively insensitive to the value of the surplus cost parameter.
2. The cost advantages obtained from the two-period model represent only a small improvement over the one-period system.
3. The difficulties in using the two-period model caused by its complexity as well as in obtaining accurate forecast data for the second quarter reduce that model's effectiveness.

VI. CONCLUSIONS/RECOMMENDATIONS

A. DISCUSSION

The basic purpose of this thesis has been to consider projected production information spanning two periods in computing RSS inventory quantities. One important assumption used throughout this thesis is that an order review is conducted at the beginning of each period for all RSS items, and that, consequently, there is no fixed ordering charge associated with any given buy. Various two-period models based on different demand probability distributions have been derived and analyzed, including the dual-period binomial system which most closely represents the situation applicable to a NARF. In this chapter, several general conclusions are drawn from the information previously presented. Additionally, recommendations for potential future actions in this area are made.

B. CONCLUSIONS

The principal results obtained from considering the application of a dual-period binomial inventory model to a NARF are listed as follows:

1. The model identifies a critical inventory quantity which can be compared to the previous period closing balance to determine both whether additional stock is needed and the amount of the stock order if replenishment action is necessary.

2. The optimal inventory quantity obtained from the two-period model is always greater than or equal to the y* value determined from a one-period system. As a result, there are expected cost benefits to be derived from using a two-period approach.

3. The two-period situation is more difficult to evaluate than the single period since expected optimal results from the second period must be used in conjunction with anticipated first-period demand to determine y*.

4. The difference between the expected optimal TVC results obtained from one and two-period inventory models is relatively small. This difference is further reduced by the consideration that the production schedule forecast will generally be less accurate for the second period than it will be for the first.

5. A fairly accurate estimate of the stockout cost rate is an essential element in the dual-period formulation. This is especially true for high cost items where the unit price will be relatively close to realistic stockout cost values. On the other hand, the results obtained from the dual-period binomial model are not particularly sensitive to changes in the surplus cost rate.

C. RECOMMENDATIONS

Several recommendations for future actions can be identified as a result of the conclusions made in this thesis. First, the sensitivity of model results to changes in the stockout cost rate indicates that the NARF should attempt to quantify all costs incurred as a result of work stoppages so that an accurate estimate of P can be made. This is an essential requirement for identifying the optimal quantity of a repair part to stock at a particular point in time, and is necessary whether a single or dual-period inventory model is used in the RSS.

The second recommendation is derived from the conclusion that the single-period model provides results which are nearly equivalent to those obtained from the dual-period system in terms of the expected TVC incurred while using an objective function which is much easier to evaluate. Additionally, the fact that production schedule forecasts for the second period are probably less accurate than for the first increases the appeal of the single-period approach. Accordingly, it is recommended that the single-period binomial inventory model as discussed in Chapter II of this thesis be used by the supporting NSC to determine RSS stock quantities for the NARF instead of the dual-period model, pending receipt of the results of additional analysis as recommended in the following paragraph.

Finally, the availability of workload forecasts beyond the second period must be considered. Although the cost advantages obtained from the two-period model over the single-period system are small, the possibility exists that these benefits are increased by incorporating this additional production information into the model. Therefore, it is recommended that the two-period model be expanded to consider projected rework data spanning up to a procurement leadtime in length, and that the optimal results from such a model be compared to the single-period results.

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